

Three Point Functions on the Sphere of Calabi-Yau d -Folds

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ABSTRACT

Using mirror symmetry in Calabi-Yau manifolds M , three point functions of $A(M)$ -model operators on the genus 0 Riemann surface in cases of one-parameter families of d -folds realized as Fermat type hypersurfaces embedded in weighted projective spaces and a two-parameter family of d -fold embedded in a weighted projective space $\mathbf{P}_{d+1}[2, 2, 2, \dots, 2, 2, 1, 1](2(d+1))$ are studied. These three point functions $\langle \mathcal{O}_a^{(1)} \mathcal{O}_b^{(l-1)} \mathcal{O}_c^{(d-l)} \rangle$ are expanded by indeterminates $q_l = e^{2\pi i t_l}$ associated with a set of Kähler coordinates $\{t_l\}$ and their expansion coefficients count the number of maps with a definite degree which map each of three points $0, 1$, and ∞ on the world sheet on some homology cycle of M associated with a cohomology element. From these analyses, we can read fusion structure of Calabi-Yau $A(M)$ -model operators. In our cases they constitute a subring of a total quantum cohomology ring of the $A(M)$ -model operators. In fact we switch off all perturbation operators on the topological theories except for marginal ones associated with Kähler forms of M . For that reason, the charge conservation of operators turns out to be a classical one. Furthermore because their first Chern classes c_1 vanish, their topological selection rules do not depend on the degree of maps, (especially a nilpotent property of operators $\mathcal{O}^{(1)} \mathcal{O}^{(d)} = 0$ is satisfied). Then these fusion couplings $\{\kappa_l\}$ are represented as some series adding up all degrees of maps.

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1 Introduction

When one considers correlation functions of an $N=2$ non-linear sigma model with a Calabi-Yau target space, they depend not only on the moduli space of the Riemann surface, but also on the properties of the target Calabi-Yau manifold (especially on the Calabi-Yau moduli spaces). There are two quasi-topological field theories (the A-model and the B-model) [1] obtained by twisting the $N=2$ non-linear sigma model [2, 3], which describe two distinct Calabi-Yau moduli spaces (a Kähler structure moduli space and a complex structure moduli space, respectively). As is well-known, the correlators of the B-model receive no quantum corrections [1, 4, 5, 6], but the correlation functions of the A-model have non-perturbative corrections which originate in holomorphic maps from the Riemann surface to the Calabi-Yau target space.

Recently by the discovery of the mirror symmetry [7, 8] between the A(M)-model and the B(W)-model for the mirror pairs (M, W) , it is becoming possible to obtain the A(M)-model correlators from the B(W)-model ones indirectly but in a form involving all the quantum corrections [5, 9]. Now it may fairly be said that the analyses of the Calabi-Yau 3-folds under the mirror symmetries have been established [10, 11, 12, 13, 14, 15, 16, 17, 18]. In this article, we study the A(M)-model correlators of some d -dimensional Calabi-Yau manifolds as mathematical physics applications under the mirror symmetry furthermore [19, 20, 21].

Because both the A-model and the B-model are pseudo-topological theories, they are characterized by their two point functions and three point functions which play important roles as the constituent blocks in these models. The two point functions of the A-model are called the topological metrics and receive no quantum corrections [1]. On the other hand, the three point functions of the A-model have non-perturbative quantum corrections and have information about the fusion structure of the observables (the classical cohomology structure and quantum corrections). Because various physical quantities are determined by this fusion structure of these operators, it is important to study the properties of the operator products (the commutativity, associativity, and the existence of the unit operator) or the factorization of the multi-point functions.

In this article, we take a genus zero Riemann surface as a world sheet, (complex) d -dimensional Calabi-Yau target spaces and analyze the properties of the three point functions under the mirror symmetry in order to clarify the effects of the non-perturbative instanton corrections.

2 The Calabi-Yau d-folds

We consider d-dimensional Calabi-Yau manifolds M in two cases;

- Case I (one-parameter families)

$$(I-1) \quad M; \mathbf{p} = X_1^{d+2} + X_2^{d+2} + \cdots + X_{d+2}^{d+2}$$

$$-(d+2)\psi(X_1 X_2 \cdots X_{d+2}) = 0 \text{ in } CP^{d+1} \text{ ,}$$

$$(I-2) \quad M; \mathbf{p} = X_1^2 + X_2^{2(d+1)} + \cdots + X_{d+2}^{2(d+1)}$$

$$-2(d+1)\psi(X_1 X_2 \cdots X_{d+2}) = 0 \text{ in } \mathbf{P}_{d+1}[d+1, \underbrace{1, 1, \dots, 1}_{(d+1) \text{ times}}](2(d+1)) \text{ ,}$$

$$(I-0) \quad M; \mathbf{p} = X_1^{l_1} + X_2^{l_2} + \cdots + X_{d+2}^{l_{d+2}}$$

$$-D\psi(X_1 X_2 \cdots X_{d+2}) = 0 \text{ in } \mathbf{P}_{d+1}[w_1, w_2, \dots, w_{d+2}](D) \text{ ,}$$

$$D := \sum_{i=1}^{d+2} w_i \text{ , } l_i := \frac{D}{w_i} \text{ , } w_{d+2} := 1 \text{ ,}$$

- Case II (two-parameter family)

$$(II) \quad M; \mathbf{p} = X_1^{d+1} + X_2^{d+1} + \cdots + X_d^{d+1} + X_{d+1}^{2(d+1)} + X_{d+2}^{2(d+1)} - 2\phi(X_{d+1} X_{d+2})^{d+1}$$

$$-2(d+1)\psi(X_1 X_2 \cdots X_{d+2}) = 0 \text{ in } \mathbf{P}_{d+1}[\underbrace{2, 2, \dots, 2}_{d \text{ times}}, 1, 1](2(d+1)) \text{ .}$$

The parameters ψ and ϕ control the deformation of the complex structures. In case (I-0), the sets of integers $(l_1, l_2, \dots, l_{d+2})$ are given in some lower dimensional cases;

- $d = 3$

d=3	l_1	l_2	l_3	l_4	l_5
(1)	5	5	5	5	5
(2)	3	6	6	6	6
(3)	2	8	8	8	8
(4)	2	5	10	10	10

- $d = 4$

d=4	l_1	l_2	l_3	l_4	l_5	l_6
(1)	2	10	10	10	10	10
(2)	6	6	6	6	6	6

- $d = 5$

d=5	l_1	l_2	l_3	l_4	l_5	l_6	l_7
(1)	4	8	8	8	8	8	8
(2)	3	9	9	9	9	9	9
(3)	2	12	12	12	12	12	12
(4)	3	4	12	12	12	12	12
(5)	2	7	14	14	14	14	14
(6)	7	7	7	7	7	7	7

Because the case (I-0) contains cases (I-1) and (I-2), we analyze cases (I-0) and (II) in the following. Mirror partners W of these Fermat type Calabi-Yau d -folds M ((I-0), (II)) are given as orbifolds divided by some maximally invariant discrete groups G ,

$$W ; \{p = 0\} / G .$$

When one thinks about the Hodge structure of the G -invariant part of the cohomology group $H^d(W)$, Hodge numbers of W are written;

$$\begin{aligned} \text{(I-0)} ; h^{d,0} = h^{d-1,1} = \dots = h^{1,d-1} = h^{0,d} = 1 , \\ \text{(II)} ; h^{d,0} = h^{0,d} = 1 , h^{d-1,1} = h^{d-2,2} = \dots = h^{2,d-2} = h^{1,d-1} = 2 . \end{aligned}$$

3 The One-Parameter Families

Firstly we investigate the case (I-0). The deformation of the complex structure of W is controlled by the structure of the Hodge decomposition of $H^d(W)$ and the information of the decomposition is given by the period matrix P of W . The period matrix is defined by using homology d -cycles $\gamma_j \in H_d(W)$ and cohomology elements $\alpha_i \in \mathcal{F}^{d-i} = H^{d,0} \oplus H^{d-1,1} \oplus \dots \oplus H^{d-i,i}$ and its matrix elements P_{ij} ($0 \leq i \leq d$, $0 \leq j \leq d$) are expressed as,

$$P_{ij} := \int_{\gamma_j} \alpha_i .$$

Especially the $\alpha_0 = \Omega$ is a globally defined nowhere-vanishing holomorphic d -form of W and can be expressed for the Fermat type hypersurface \mathbf{p} by [16, 22, 23],

$$\begin{aligned} \Omega &:= \int_{\gamma} \frac{d\mu}{\mathbf{p}} , \\ d\mu &:= \sum_{a=1}^{d+2} (-1)^{a-1} w_a X_a dX_1 \wedge dX_2 \wedge \dots \wedge \widehat{dX_a} \wedge \dots \wedge dX_{d+2} , \end{aligned}$$

where γ is a small one-dimensional cycle winding around the hypersurface defined as a zero locus of \mathbf{p} ,

$$Z_p := \{(X_1, X_2, \dots, X_{d+2}) ; \mathbf{p} = 0\} .$$

Also the α_i 's are defined as,

$$\begin{aligned} \alpha_i &:= \Theta_z^i \Omega , \quad \Theta_z := z \frac{d}{dz} , \\ z &:= (D\psi)^{-D} . \end{aligned}$$

Now we pick the elements of the period matrix P in the zero-th row,

$$P_{0j} = \int_{\gamma_j} \Omega \equiv \varpi_j .$$

Using the explicit formula of the d-form Ω , we obtain a differential equation satisfied by ϖ_j 's,

$$\left[\Theta_z^{d+1} - \gamma \prod_{l=1}^{D-1} \left(\Theta_z + \frac{l}{D} \right) \right] \varpi_j = 0 \quad , \quad \gamma := \frac{D^D}{\prod_{i=1}^{d+2} w_i^{w_i}} \quad , \quad (1)$$

where the product $\prod_{l=1}^{D-1}$ means that the variable l runs over integers ranging from one to $(D-1)$ which are not divisible by any l_i ,

$$\prod_{l=1}^{D-1} := \prod_{\substack{l=1 \\ l_i \nmid l}}^{D-1} \quad .$$

We solve this equation (1) in a series form,

$$P_{0j} = \varpi_j(z) := \sum_{l=0}^j \frac{1}{l!} \left(\frac{\log z}{2\pi i} \right)^l \times \sum_{m=0}^{\infty} b_{j-l,m} \cdot z^m \quad ,$$

$$b_{n,m} := \frac{1}{n!} \left(\frac{1}{2\pi i} \cdot \frac{\partial}{\partial \rho} \right)^n \left\{ \frac{\Gamma(D(m+\rho)+1)}{\Gamma(D\rho+1)} \cdot \prod_{i=1}^{d+2} \frac{\Gamma(w_i\rho+1)}{\Gamma(w_i(m+\rho)+1)} \right\} \Big|_{\rho=0} \quad ,$$

and construct the period matrix P of W ,

$$P_{ij} = \int_{\gamma_j} \Theta_z^i \Omega = \Theta_z^i \varpi_j \quad .$$

In this process, we do not construct homology d-cycles explicitly. In order to study properties of these homology d-cycles $\{\gamma_0, \gamma_1, \dots, \gamma_d\}$, we perform a monodromy transformation T about the point $z = 0$,

$$T; z \rightarrow e^{2\pi i} z \quad .$$

Then the period matrix P changes into a form,

$$P \rightarrow P \cdot A \quad , \quad A = \exp(N) \quad ,$$

$$N := \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & 0 & 1 & \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix} \quad , \quad (N^{d+1} = 0) \quad ,$$

and we can read the action of the monodromy transformation T on the set of cycles $(\gamma_0 \gamma_1 \dots \gamma_d)$, which we use implicitly in defining the period matrix P ,

$$T(\gamma_0 \gamma_1 \dots \gamma_d) = (\gamma_0 \gamma_1 \dots \gamma_d) A \quad .$$

A peculiar property $N^{d+1} = 0$ of the above monodromy matrix $A = \exp(N)$ indicates the maximally unipotent monodromy condition of these homology cycles at the point $z = 0$ and is consistent with the usual mirror conjecture. When one studies the variation of the complex structure of W , one deforms cohomology elements but fixes homology cycles. So we investigate the set of the cohomology classes $(\alpha_0, \alpha_1, \dots, \alpha_d) = (\Omega, \Theta_z \Omega, \dots, \Theta_z^d \Omega)$. The cohomology group elements $\alpha_l = \Theta_z^l \Omega$ are expressed as,

$$\Theta_z^l \Omega = \sum_{m=1}^l \int \frac{\varphi_{lm}}{\mathbf{p}^{m+1}} d\mu \ ,$$

$$\varphi_{lm} = \left\{ \sum_{k=1}^m (-1)^{m-k} \cdot {}_m C_k \cdot \left(\frac{-k}{D} \right)^l \right\} \times (D\psi)^m \cdot (X_1 X_2 \cdots X_{d+2})^m \ .$$

When one deforms the complex structure of W , the set of cohomology classes is modified and the structure of the Hodge decomposition $\oplus_{p=0}^d H^{d-p,p}(W)$ undergoes a change. In order to look into the variation of the complex structure of W , we change the period matrix P into an upper triangular one Φ with unit diagonal elements by the sweeping-out method. In this operation, the set of homology d-cycles $(\gamma_0 \gamma_1 \cdots \gamma_d)$ remains unchanged, but the cohomology basis $(\alpha_0 \alpha_1 \cdots \alpha_d)$ turns into a new basis $(\tilde{\alpha}_0 \tilde{\alpha}_1 \cdots \tilde{\alpha}_d)$,

$$\tilde{\alpha}_0 := \frac{1}{\varpi_0} \alpha_0 \in H^{d,0} \ ,$$

$$\tilde{\alpha}_l := \frac{1}{\tilde{\kappa}_{l-1}} \Theta_z \frac{1}{\tilde{\kappa}_{l-2}} \Theta_z \cdots \Theta_z \frac{1}{\tilde{\kappa}_1} \Theta_z \frac{1}{\tilde{\kappa}_0} \Theta_z \left(\frac{\alpha_0}{\varpi_0} \right) \in \mathcal{F}^{d-l} \ , \ (1 \leq l \leq d) \ ,$$

$$\tilde{\kappa}_0 := \Theta_z \left(\int_{\gamma_1} \tilde{\alpha}_0 \right) = \Theta_z \omega_1 \ ,$$

$$\tilde{\kappa}_m := \Theta_z \left(\int_{\gamma_{m+1}} \tilde{\alpha}_m \right) = \Theta_z \frac{1}{\tilde{\kappa}_{m-1}} \Theta_z \frac{1}{\tilde{\kappa}_{m-2}} \Theta_z \cdots \Theta_z \frac{1}{\tilde{\kappa}_1} \Theta_z \frac{1}{\tilde{\kappa}_0} \Theta_z \omega_{m+1} \ , \ (1 \leq m \leq d-1) \ ,$$

$$\omega_n := \frac{\varpi_n}{\varpi_0} \ .$$

Using this new basis $\{\tilde{\alpha}_i\}$, we can write down the resulting period matrix Φ ,

$$\Phi_{ij} := \int_{\gamma_j} \tilde{\alpha}_i \ ,$$

$$\Phi = \begin{pmatrix} 1 & \Phi_{01} & \Phi_{02} & \Phi_{03} & \Phi_{04} & \cdots & \Phi_{0d-2} & \Phi_{0d-1} & \Phi_{0d} \\ & 1 & \Phi_{12} & \Phi_{13} & \Phi_{14} & \cdots & \Phi_{1d-2} & \Phi_{1d-1} & \Phi_{1d} \\ & & 1 & \Phi_{23} & \Phi_{24} & \cdots & \Phi_{2d-2} & \Phi_{2d-1} & \Phi_{2d} \\ & & & 1 & \Phi_{34} & \cdots & \Phi_{3d-2} & \Phi_{3d-1} & \Phi_{3d} \\ & & & & \ddots & \ddots & \vdots & \vdots & \vdots \\ & & & & & 1 & \Phi_{d-3d-2} & \Phi_{d-3d-1} & \Phi_{d-3d} \\ & & & & & & 1 & \Phi_{d-2d-1} & \Phi_{d-2d} \\ & & & & & & & 1 & \Phi_{d-1d} \\ O & & & & & & & & 1 \end{pmatrix} \ .$$

To investigate the complex structure of W , we think about a differential equation of P ,

$$\begin{aligned} \Theta_z P(z) &= \tilde{C}_z P(z) \ , \\ \tilde{C}_z &:= \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & 0 & 1 & \\ & & & & 0 & 1 \\ \sigma_{N-1} & \sigma_{N-2} & \sigma_{N-3} & \cdots & \sigma_2 & \sigma_1 \end{pmatrix} \ , \\ \sigma_m &:= \frac{\gamma z}{1 - \gamma z} \sum'_{1 \leq n_1 < n_2 < \cdots < n_m \leq D-1} \frac{n_1}{D} \cdot \frac{n_2}{D} \cdots \frac{n_m}{D} \ , \end{aligned}$$

where the sum $\sum'_{1 \leq n_1 < n_2 < \cdots < n_m \leq D-1}$ means that the variables $\{n_i\}$ run over integers which are not divisible by any l_j ,

$$\sum'_{1 \leq n_1 < n_2 < \cdots < n_m \leq D-1} = \sum_{\substack{1 \leq n_1 < n_2 < \cdots < n_m \leq D-1 \\ l_j \nmid n_i}} \ .$$

By introducing a new variable $t := \omega_1(z)$, the above differential equation can be rewritten,

$$\begin{aligned} \partial_t \Phi(t) &= C_t \Phi(t) \ , \quad \Phi_{lm} := \int_{\gamma_m} \tilde{\alpha}_l(t) \ , \\ C_t &:= \begin{pmatrix} 0 & \kappa_0 & & & & O \\ & 0 & \kappa_1 & & & \\ & & 0 & \kappa_2 & & \\ & & & \ddots & \ddots & \\ & & & & 0 & \kappa_{d-2} \\ & & & & & 0 & \kappa_{d-1} \\ O & & & & & & 0 \end{pmatrix} \ , \\ \tilde{\alpha}_0(t) &:= \frac{1}{\varpi_0} \alpha_0 \ , \\ \tilde{\alpha}_l(t) &:= \frac{1}{\kappa_{l-1}} \partial_t \frac{1}{\kappa_{l-2}} \partial_t \cdots \partial_t \frac{1}{\kappa_1} \partial_t \frac{1}{\kappa_0} \partial_t \left(\frac{\alpha_0}{\varpi_0} \right) \ , \quad (1 \leq l \leq d) \ , \\ \kappa_0 &:= \partial_t \omega_1 = 1 \ , \quad (t \equiv \omega_1) \\ \kappa_m &:= \partial_t \frac{1}{\kappa_{m-1}} \partial_t \frac{1}{\kappa_{m-2}} \partial_t \cdots \partial_t \frac{1}{\kappa_1} \partial_t \frac{1}{\kappa_0} \partial_t \omega_{m+1} \ , \quad (1 \leq m \leq d-1) \ . \end{aligned}$$

From this equation, we can read the action of the differential operator ∂_t on the cohomology basis $\tilde{\alpha}_l(t)$,

$$\begin{aligned} \partial_t \tilde{\alpha}_{j-1}(t) &= \kappa_{j-1}(t) \tilde{\alpha}_j(t) \ , \quad (1 \leq j \leq d) \ , \\ \partial_t \tilde{\alpha}_d(t) &= 0 \ , \\ \kappa_{j-1}(t) &= \partial_t \Phi_{j-1,j} \ , \quad (1 \leq j \leq d) \ . \end{aligned}$$

When we define two point functions,

$$\langle \tilde{\alpha}_i \tilde{\alpha}_j \rangle := \int_W \tilde{\alpha}_i \wedge \tilde{\alpha}_j \quad ,$$

they satisfy relations,

$$\langle \tilde{\alpha}_i \tilde{\alpha}_j \rangle := \delta_{i+j,d} \langle \gamma_i^* \gamma_j^* \rangle \quad ,$$

where the set $\{\gamma_m^*\}$ is the dual of the homology d-cycles $\{\gamma_m\}$,

$$\int_{\gamma_m} \gamma_n^* = \delta_{m,n} \quad .$$

We translate these relations in the above B(W)-model into the operator structures of the corresponding A(M)-model,

$$\begin{aligned} \mathcal{O}^{(1)} \mathcal{O}^{(j-1)} &= \kappa_{j-1}(t) \mathcal{O}^{(j)} \quad , \quad (1 \leq j \leq d) \quad , \\ \mathcal{O}^{(1)} \mathcal{O}^{(d)} &= 0 \quad , \\ \langle \mathcal{O}^{(i)} \mathcal{O}^{(j)} \rangle &= \delta_{i+j,d} \eta_{ij} \quad , \end{aligned}$$

for A(M)-model operators $\mathcal{O}^{(i)} \in H^{i,i}(M)$, $(1 \leq i \leq d)$. The above operator product structure of the A(M)-model observables is meaningful when one defines correlation functions in the following way,

$$\begin{aligned} \langle \mathcal{O}^{(1)} \mathcal{O}^{(j-1)} \dots \rangle &:= \int \mathcal{D}[X, \chi, \rho] \mathcal{O}^{(1)} \mathcal{O}^{(j-1)} \dots e^{-L_A} \quad , \\ L_A &:= t \int_{\Sigma} X^*(e) + \int_{\Sigma} d^2 z \{Q, V\} \quad , \end{aligned}$$

where Q is a BRST charge of the A(M)-model and V is given as,

$$V := it g_{i\bar{j}} (\rho_z^{\bar{j}} \partial_{\bar{z}} X^i + \rho_{\bar{z}}^j \partial_z X^{\bar{i}}) \quad .$$

Also the integral,

$$\int_{\Sigma} X^*(e) = \int_{\Sigma} g_{i\bar{j}} (\partial_z X^i \partial_{\bar{z}} X^{\bar{j}} - \partial_{\bar{z}} X^i \partial_z X^{\bar{j}}) dz \wedge d\bar{z} \quad ,$$

is the pullback of the Kähler form e of M and equals to the degree of the maps X . (This operator is a 2-form version on the world sheet of the local observable $\mathcal{O}^{(1)}$). When we view this A(M)-model as a deformed theory from some topological field theory, it is characteristic that we perturb the original topological theory by adding only operators associated with Kähler forms of M . As for the (topological) selection rule for the A(M)-model correlators, it depends on the degree of maps X generally because the virtual dimension (the ghost number anomaly) is given as,

$$vir\dim = (\dim M) \cdot (1 - g) + \int_{\Sigma} X^* c_1(M) \quad ,$$

where g is the genus of the Riemann surface and $c_1(M)$ is the first Chern class of M . However for specific cases $c_1 = 0$ (Calabi-Yau cases), the virtual dimension is independent of the degree of maps and then the selection rule does not depend on the degree of maps. Collecting these considerations, we can understand that the degree conservation of $A(M)$ -model operators in each fusion coincides with a classical one for Calabi-Yau cases. (Especially a relation $\mathcal{O}^{(1)}\mathcal{O}^{(d)} = 0$ ($d := \dim M$) is satisfied). On the other hand, when we expand fusion couplings $\{\kappa_l\}$ of operators $\mathcal{O}^{(1)}$ and $\mathcal{O}^{(l)}$ with respect to an indeterminate $q := e^{2\pi it}$, they contain all non-negative powers of q generally because the selection rule of observables is independent of the degree of maps.

Next let us consider a moduli space of the Riemann surface (world sheet). The dimension of a moduli space $\mathcal{M}_{g,s}$ of a genus g Riemann surface with s punctures is given as,

$$\dim \mathcal{M}_{g,s} = 3(g - 1) + s \ .$$

Because we consider three point couplings ($s = 0$) on the sphere ($g = 0$), the degree of the world sheet moduli comes 3 from the positions of the operator insertions and -3 from the $SL(2, \mathbf{C})$ -invariance of the CP^1 respectively. Adding up these two contributions, we obtain the dimension of the moduli space $\mathcal{M}_{0,3}$ of the Riemann surface,

$$\dim \mathcal{M}_{0,3} = 3 - 3 = 0 \ .$$

Judging from this counting of the dimension only, we cannot understand whether our systems (the Calabi-Yau matters) couple with the topological gravity or not. In fact in our cases the systems do not couple the gravity because we fix the positions of the operator insertions and do not move them.

Using this translation, we obtain three point functions $\kappa_l(t)$ of the $A(M)$ -model explicitly,

$$\begin{aligned} \kappa_0 &= 1 \ , \\ \kappa_l &= \partial_t \frac{1}{\kappa_{l-1}} \partial_t \frac{1}{\kappa_{l-2}} \partial_t \cdots \partial_t \frac{1}{\kappa_2} \partial_t \frac{1}{\kappa_1} \partial_t \frac{1}{\kappa_0} \partial_t S_{l+1}(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_{l+1}) \ , \ (1 \leq l \leq d-1) \ , \\ \tilde{x}_n &:= \frac{1}{n!} D_\rho^n \log \tilde{\omega}_0(z; \rho) \Big|_{\rho=0} \ , \ D_\rho := \frac{1}{2\pi i} \cdot \frac{\partial}{\partial \rho} \ , \\ \tilde{\omega}_0(z, \rho) &:= \sum_{m=0}^{\infty} \frac{\Gamma(D(m+\rho)+1)}{\Gamma(D\rho+1)} \cdot \prod_{i=1}^{d+2} \frac{\Gamma(w_i\rho+1)}{\Gamma(w_i(m+\rho)+1)} z^{m+\rho} \ , \end{aligned}$$

where the function " S_n " is the Schur function defined as the coefficients in the following expansion,

$$\sum_{n=0}^{\infty} S_n(x_1, x_2, \cdots, x_n) u^n := \exp \left(\sum_{m=1}^{\infty} x_m u^m \right) \ .$$

We write down expressions of these couplings κ_l in a series with respect to a parameter $q := e^{2\pi i t}$, $t = S_1(\tilde{x}_1) = \tilde{x}_1$,

$$\begin{aligned}\kappa_l &= 1 + \alpha_l q + O(q^2) , \\ \alpha_l &= \frac{D!}{\prod_{i=1}^{d+2} w_i!} \times \left[\tilde{A}_{d+1-l} - \left(\sum_{l=1}^D \frac{D}{l} - \sum_{i=1}^{d+2} \sum_{m_i=1}^{w_i} \frac{w_i}{m_i} \right) \right] , \\ \tilde{A}_m &:= \sum'_{1 \leq m_1 < m_2 < \dots < m_n \leq D-1} \frac{D-m_1}{m_1} \cdot \frac{D-m_2}{m_2} \cdot \dots \cdot \frac{D-m_n}{m_n} .\end{aligned}$$

Then a d-point coupling of d operators $\mathcal{O}^{(1)}$'s can be calculated up to an overall normalization factor,

$$\begin{aligned}\langle \mathcal{O}^{(1)} \mathcal{O}^{(1)} \dots \mathcal{O}^{(1)} \rangle &= \kappa_1 \kappa_2 \dots \kappa_{d-2} \eta_{1,d-1} \\ &= 1 + \left\{ \frac{D^D}{\prod_{i=1}^{d+2} w_i^{w_i}} - 2 \cdot \frac{D!}{\prod_{i=1}^{d+2} w_i!} \right. \\ &\quad \left. - d \cdot \frac{D!}{\prod_{i=1}^{d+2} w_i!} \times \left[\sum_{l=1}^D \frac{D}{l} - \sum_{i=1}^{d+2} \sum_{m_i=1}^{w_i} \frac{w_i}{m_i} \right] \right\} \cdot q + O(q^2) .\end{aligned}$$

4 The Two-Parameter Family

Secondly let us investigate the model (II). The G -invariant parts of the Hodge structure of the mirror manifolds $W = M/G$ are characterized by a set of homology d-cycles,

$$\{\gamma_0, \gamma_1^{(1)}, \gamma_1^{(2)}, \dots, \gamma_{d-1}^{(1)}, \gamma_{d-1}^{(2)}, \gamma_d\} ,$$

and a set of cohomology elements of W ,

$$\begin{aligned}\{\alpha_0, \alpha_1^{(1)}, \alpha_1^{(2)}, \dots, \alpha_{d-1}^{(1)}, \alpha_{d-1}^{(2)}, \alpha_d\} , \\ \alpha_i^{(\bullet)} \in \mathcal{F}^{d-i} = H^{d,0} \oplus H^{d-1,1} \oplus \dots \oplus H^{d-i,i} .\end{aligned}$$

Especially we take the following elements as this basis,

$$\begin{aligned}\alpha_0 &:= \Omega = \int_{\gamma} \frac{d\mu}{\mathbf{p}} \quad (\text{a holomorphic d form of } W) , \\ \alpha_l^{(1)} &:= \Theta_x^l \Omega , \quad (1 \leq l \leq d-1) , \\ \alpha_l^{(2)} &:= \Theta_x^{l-1} \Theta_y \Omega , \quad (1 \leq l \leq d-1) , \\ \alpha_d &:= (\Theta_x^d - 2 \cdot \Theta_x^{d-1} \Theta_y) \Omega ,\end{aligned}$$

$$x := \frac{-2\phi}{[2(d+1)\psi]^{d+1}} , \quad y := \frac{1}{(2\phi)^2} ,$$

$$\Theta_x := x \frac{\partial}{\partial x} , \quad \Theta_y := y \frac{\partial}{\partial y} .$$

By using these elements, we can write a period matrix P of the mirror W in a block form, which contains $(d+1) \times (d+1)$ block matrices, (4 1×1 matrices, $2 \cdot (d-1)$ 1×2 matrices, $2 \cdot (d-1)$ 2×1 matrices, and $(d-1) \cdot (d-1)$ 2×2 matrices). The (l, m) -th block matrix P_{lm} of P is defined as,

$$P_{0m} := \begin{cases} \int_{\gamma_0} \alpha_0 & (m=0) \\ \left(\int_{\gamma_m^{(1)}} \alpha_0 \int_{\gamma_m^{(2)}} \alpha_0 \right) & (1 \leq m \leq d-1) \\ \int_{\gamma_d} \alpha_0 & (m=d) \end{cases} ,$$

$$= \begin{cases} \varpi_0 & (m=0) \\ (\varpi_m^{(1)} \varpi_m^{(2)}) & (1 \leq m \leq d-1) \\ \varpi_d & (m=d) \end{cases} ,$$

$$P_{lm} := \begin{cases} \begin{pmatrix} \int_{\gamma_0} \alpha_l^{(1)} \\ \int_{\gamma_0} \alpha_l^{(2)} \end{pmatrix} & (m=0) \\ \begin{pmatrix} \int_{\gamma_m^{(1)}} \alpha_l^{(1)} & \int_{\gamma_m^{(2)}} \alpha_l^{(1)} \\ \int_{\gamma_m^{(1)}} \alpha_l^{(2)} & \int_{\gamma_m^{(2)}} \alpha_l^{(2)} \end{pmatrix} & (1 \leq m \leq d-1) \\ \begin{pmatrix} \int_{\gamma_d} \alpha_l^{(1)} \\ \int_{\gamma_d} \alpha_l^{(2)} \end{pmatrix} & (m=d) \end{cases} ,$$

$$= \begin{cases} \begin{pmatrix} \Theta_x^l \varpi_0 \\ \Theta_x^{l-1} \Theta_y \varpi_0 \end{pmatrix} & (m=0) \\ \begin{pmatrix} \Theta_x^l \varpi_m^{(1)} & \Theta_x^l \varpi_m^{(2)} \\ \Theta_x^{l-1} \Theta_y \varpi_m^{(1)} & \Theta_x^{l-1} \Theta_y \varpi_m^{(2)} \end{pmatrix} & (1 \leq m \leq d-1) \\ \begin{pmatrix} \Theta_x^l \varpi_d \\ \Theta_x^{l-1} \Theta_y \varpi_d \end{pmatrix} & (m=d) \end{cases} \quad (l=1, 2, \dots, d-1) ,$$

$$P_{dm} := \begin{cases} \int_{\gamma_0} \alpha_d & (m=0) \\ \left(\int_{\gamma_m^{(1)}} \alpha_d \int_{\gamma_m^{(2)}} \alpha_d \right) & (1 \leq m \leq d-1) \\ \int_{\gamma_d} \alpha_d & (m=d) \end{cases}$$

$$= \begin{cases} (\Theta_x^d - 2\Theta_x^{d-1}\Theta_y) \varpi_0 & (m=0) \\ \begin{pmatrix} (\Theta_x^d - 2\Theta_x^{d-1}\Theta_y) \varpi_m^{(1)} & (\Theta_x^d - 2\Theta_x^{d-1}\Theta_y) \varpi_m^{(2)} \\ (\Theta_x^d - 2\Theta_x^{d-1}\Theta_y) \varpi_d \end{pmatrix} & (1 \leq m \leq d-1) \\ (\Theta_x^d - 2\Theta_x^{d-1}\Theta_y) \varpi_d & (m=d) \end{cases} .$$

We obtain a set of differential equations satisfied by $\varpi_m^{(\bullet)}$ ($0 \leq m \leq d$),

$$\mathcal{D}_{(1)} \varpi_m(x, y) = 0 , \quad \mathcal{D}_{(2)} \varpi_m(x, y) = 0 ,$$

$$\mathcal{D}_{(1)} := \Theta_x^{d-1} (\Theta_x - 2\Theta_y)$$

$$-(d+1)^{d+1}x(\Theta_x + \frac{d}{d+1})(\Theta_x + \frac{d-1}{d+1}) \cdots (\Theta_x + \frac{2}{d+1})(\Theta_x + \frac{1}{d+1}) , \quad (2)$$

$$\mathcal{D}_{(2)} := \Theta_y^2 - y(\Theta_x - 2\Theta_y)(\Theta_x - 2\Theta_y - 1) . \quad (3)$$

The $2d$ linear independent solutions are written down,

$$\varpi_0 := \hat{\varpi}_0(x, y; \rho_1, \rho_2) \Big|_{\rho_1=\rho_2=0} , \quad (4)$$

$$\varpi_l^{(1)} := \frac{1}{l!} \mathcal{D}_{\rho_1}^l \hat{\varpi}_0(x, y; \rho_1, \rho_2) \Big|_{\rho_1=\rho_2=0} , \quad (l = 1, 2, \dots, d-1) , \quad (5)$$

$$\varpi_l^{(2)} := \frac{1}{(l-1)!} \mathcal{D}_{\rho_1}^{l-1} \mathcal{D}_{\rho_2} \hat{\varpi}_0(x, y; \rho_1, \rho_2) \Big|_{\rho_1=\rho_2=0} , \quad (l = 1, 2, \dots, d-1) , \quad (6)$$

$$\varpi_d := \frac{1}{d!} (2 \mathcal{D}_{\rho_1}^d + d \cdot \mathcal{D}_{\rho_1}^{d-1} \mathcal{D}_{\rho_2}) \hat{\varpi}_0(x, y; \rho_1, \rho_2) \Big|_{\rho_1=\rho_2=0} , \quad (7)$$

where

$$\begin{aligned} \hat{\varpi}_0(x, y, \rho_1, \rho_2) &:= \sum_{m,n \geq 0} \frac{\Gamma((d+1)(m+\rho_1)+1)}{\Gamma((d+1)\rho_1+1)} \\ &\times \left[\frac{\Gamma(1+\rho_1)}{\Gamma(m+1+\rho_1)} \right]^d \times \left[\frac{\Gamma(1+\rho_2)}{\Gamma(n+1+\rho_2)} \right]^2 \\ &\times \left[\frac{\Gamma(\rho_1-2\rho_2+1)}{\Gamma(m-2n+\rho_1-2\rho_2+1)} \right] x^{m+\rho_1} y^{n+\rho_2} . \\ \mathcal{D}_{\rho_i} &:= \frac{1}{2\pi i} \frac{\partial}{\partial \rho_i} . \end{aligned}$$

The property of the set of homology cycles is characterized by monodromy transformations around the points $x = 0$ or $y = 0$. When we turn around the points $x = 0$ or $y = 0$,

$$x \rightarrow e^{2\pi i} x \text{ or } y \rightarrow e^{2\pi i} y ,$$

the period matrix P of W is transformed into a form,

$$P_{lm} \rightarrow \sum_n P_{ln} T_{nm}^{(i)} , \quad \begin{cases} i = 1 & (x = 0) \\ i = 2 & (y = 0) \end{cases} ,$$

where the $2d \times 2d$ matrices $T^{(1)}$ and $T^{(2)}$ are expressed as,

$$\begin{cases} T^{(1)} = \exp(N^{(1)}) & (\text{around } x = 0) \\ T^{(2)} = \exp(N^{(2)}) & (\text{around } y = 0) \end{cases} ,$$

$$N^{(1)} := \begin{matrix} & \overbrace{1} & \overbrace{2} & \overbrace{2} & \overbrace{2} & \cdots & \overbrace{2} & \overbrace{1} \\ \begin{matrix} 1\{ \\ 2\{ \\ 2\{ \\ \vdots \\ 2\{ \\ 2\{ \\ 1\{ \end{matrix} & \begin{pmatrix} 0 & e_1^{(1)} & & & & & O \\ & 0 & I & & & & \\ & & 0 & I & & & \\ & & & \ddots & \ddots & & \\ & & & & 0 & I & \\ & O & & & & 0 & e_1^{(2)} \\ & & & & & & 0 \end{pmatrix} & , \end{matrix} \quad (8)$$

$$N^{(2)} := \begin{matrix} & \overbrace{1} & \overbrace{2} & \overbrace{2} & \overbrace{2} & \dots & \overbrace{2} & \overbrace{1} \\ \begin{matrix} 1\{ \\ 2\{ \\ 2\{ \\ \vdots \\ 2\{ \\ 2\{ \\ 1\{ \end{matrix} & \begin{pmatrix} 0 & e_2^{(1)} & & & & & O \\ & 0 & I' & & & & \\ & & 0 & I' & & & \\ & & & \ddots & \ddots & & \\ & & & & 0 & I' & \\ & & & & & 0 & e_2^{(2)} \\ O & & & & & & 0 \end{pmatrix} \end{matrix} , \quad (9)$$

where the matrices in the blocks are given as,

$$\begin{aligned} e_1^{(1)} &= \begin{pmatrix} 1 & 0 \end{pmatrix} , & e_1^{(2)} &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} , \\ e_2^{(1)} &= \begin{pmatrix} 0 & 1 \end{pmatrix} , & e_2^{(2)} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \\ I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , & I' &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} . \end{aligned}$$

A simple calculation shows several relations,

$$\begin{aligned} \{N^{(1)}\}^{d+1} &= 0 , \quad \{N^{(2)}\}^2 = 0 , \quad \{N^{(1)}\}^d - 2 \cdot \{N^{(1)}\}^{d-1} \cdot \{N^{(2)}\} = 0 , \\ [N^{(1)}, N^{(2)}] &= 0 , \end{aligned}$$

and we obtain an evidence of the maximally unipotent monodromy (the nilpotent properties $(N^{(1)})^{d+1} = 0$, $(N^{(2)})^2 = 0$ show that maximally unipotent monodromy is realized at the point $(x, y) = (0, 0)$) and the homology cycles seem to be chosen appropriately. Let us introduce mirror maps in the model (II),

$$t(x, y) := \frac{\varpi_1^{(1)}}{\varpi_0} , \quad s(x, y) := \frac{\varpi_1^{(2)}}{\varpi_0} . \quad (10)$$

These maps behave under the transformations around $x = 0$ or $y = 0$ as,

$$\begin{aligned} t(e^{2\pi i} x, y) &= t(x, y) + 1 , \quad t(x, e^{2\pi i} y) = t(x, y) , \\ s(x, e^{2\pi i} y) &= s(x, y) + 1 , \quad s(e^{2\pi i} x, y) = s(x, y) . \end{aligned}$$

Because of these properties, one may regard these maps as the coordinates of the Kähler moduli space, which are defined modulo some integer shifts in the physical situations. Also the elements in the zero-th row of the period matrix Φ are given by the following functions,

$$\begin{aligned} \omega_0 &= 1 , \\ \omega_l^{(1)} &\equiv \frac{\varpi_l^{(1)}}{\varpi_0} = S_l(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_l) \end{aligned}$$

$$\begin{aligned}
&= \sum_{m=0}^l a_{l-m} \frac{t^m}{m!} , \quad (l = 1, 2, \dots, d-1) , \\
\omega_l^{(2)} &\equiv \frac{\varpi_l^{(2)}}{\varpi_0} = s \cdot S_{l-1}(\tilde{x}_1 + \tilde{y}_1, \tilde{x}_2 + \tilde{y}_2, \dots, \tilde{x}_{l-1} + \tilde{y}_{l-1}) \\
&= \sum_{m=1}^l a_{l-m} \frac{t^{m-1} \cdot s}{(m-1)!} + \sum_{m=2}^l c_{l-m} \frac{t^{m-2}}{(m-2)!} , \quad (l = 1, 2, \dots, d-1) , \\
\omega_d &\equiv \frac{\varpi_d}{\varpi_0} = 2 \cdot S_d(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_d) + s \cdot S_{d-1}(\tilde{x}_1 + \tilde{y}_1, \tilde{x}_2 + \tilde{y}_2, \dots, \tilde{x}_{d-1} + \tilde{y}_{d-1}) , \\
\begin{cases} \tilde{x}_m &:= \frac{1}{m!} \mathcal{D}_{\rho_1}^m \log \hat{\omega}_0(x, y; \rho_1, \rho_2) \Big|_{\rho_1=\rho_2=0} , \\ \tilde{y}_m &:= \frac{1}{m!} \mathcal{D}_{\rho_1}^m \log [\mathcal{D}_{\rho_2} \hat{\omega}_0(x, y; \rho_1, \rho_2)] \Big|_{\rho_1=\rho_2=0} , \\ \hat{y}_m &:= \frac{1}{m!} \mathcal{D}_{\rho_1}^m \mathcal{D}_{\rho_2} \log \hat{\omega}_0(x, y; \rho_1, \rho_2) \Big|_{\rho_1=\rho_2=0} , \end{cases} \\
a_k &:= \sum_{n \geq 1} \sum_{\{\# 1\}} \frac{1}{n!} \tilde{x}_{m_1} \tilde{x}_{m_2} \cdots \tilde{x}_{m_n} , \quad c_k := \sum_{n \geq 0} \sum_{\{\# 2\}} \frac{1}{n!} \tilde{x}_{m_1} \tilde{x}_{m_2} \cdots \tilde{x}_{m_n} \hat{y}_{m_{n+1}} , \\
\{\# 1\} &:= \{m_1 \geq 2, m_2 \geq 2, \dots, m_n \geq 2, m_1 + m_2 + \cdots + m_n = k\} , \\
\{\# 2\} &:= \{m_1 \geq 2, m_2 \geq 2, \dots, m_n \geq 2, m_{n+1} \geq 1, m_1 + m_2 + \cdots + m_n + m_{n+1} = k+1\} .
\end{aligned}$$

Next multiplying some block lower triangular matrix to the period matrix P from the left, we obtain a block upper triangular period matrix Φ which has unit matrices in its block diagonal parts,

$$\Phi = \begin{pmatrix} 1 & \omega_1^{(\bullet)} & \omega_2^{(\bullet)} & \omega_3^{(\bullet)} & \omega_4^{(\bullet)} & \cdots & \omega_{d-2}^{(\bullet)} & \omega_{d-1}^{(\bullet)} & \omega_d \\ & I & \widehat{M}_2^{(1)} & \widehat{M}_3^{(1)} & \widehat{M}_4^{(1)} & \cdots & \widehat{M}_{d-2}^{(1)} & \widehat{M}_{d-1}^{(1)} & u^{(1)} \\ & & I & \widehat{M}_3^{(2)} & \widehat{M}_4^{(2)} & \cdots & \widehat{M}_{d-2}^{(2)} & \widehat{M}_{d-1}^{(2)} & u^{(2)} \\ & & & I & \widehat{M}_4^{(3)} & \cdots & \widehat{M}_{d-2}^{(3)} & \widehat{M}_{d-1}^{(3)} & u^{(3)} \\ & & & & \ddots & \ddots & \vdots & \vdots & \vdots \\ & & & & & I & \widehat{M}_{d-2}^{(d-3)} & \widehat{M}_{d-1}^{(d-3)} & u^{(d-3)} \\ & & & & & & I & \widehat{M}_{d-1}^{(d-2)} & u^{(d-2)} \\ & & & & & & & I & u^{(d-1)} \\ O & & & & & & & & 1 \end{pmatrix} .$$

The components in the first row are defined by using the $\omega_l^{(i)}$,

$$\omega_l^{(\bullet)} := \begin{pmatrix} \omega_l^{(1)} & \omega_l^{(2)} \end{pmatrix} , \quad (l = 1, 2, \dots, d-1) .$$

The blocks $\widehat{M}_l^{(m)}$ can be written as,

$$\begin{aligned}
\widehat{M}_l^{(m)} &= U_{l-m} + \sum_{n=0}^{l-m-1} A_n^{(m)} U_{l-m-n-1} , \\
&(l > m; m = 1, 2, \dots, d-2; 2 \leq l \leq d-1) ,
\end{aligned}$$

$$U_n := \begin{pmatrix} \frac{t^n}{n!} & \frac{t^{n-1}s}{(n-1)!} \\ 0 & \frac{t^n}{n!} \end{pmatrix}, \quad (n = 1, 2, \dots),$$

$$U_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I,$$

where the matrices $A_n^{(m)}$ are defined iteratively,

$$A_n^{(1)} := \begin{pmatrix} a_{n+1} + \partial_t a_{n+2} & c_{n-1} + \partial_t c_n \\ \partial_s a_{n+2} & a_{n+1} + \partial_s c_n \end{pmatrix}, \quad (n = 1, 2, \dots),$$

$$A_0^{(1)} := \begin{pmatrix} \partial_t a_2 & \partial_t c_0 \\ \partial_s a_2 & \partial_s c_0 \end{pmatrix},$$

$$A_n^{(m+1)} := (I + \partial_t A_0^{(m)})^{-1} \cdot (A_n^{(m)} + \partial_t A_{n+1}^{(m)}), \quad (m \geq 1).$$

Also the matrices $u^{(l)}$ can be written as,

$$u^{(1)} = \begin{pmatrix} \partial_t \omega_d \\ \partial_s \omega_d \end{pmatrix},$$

$$u^{(l)} := (\partial_t \widehat{M}_l^{(l-1)})^{-1} \partial_t u^{(l-1)}, \quad (l = 2, 3, \dots, d-1).$$

Let us investigate differential equations for the resulting period Φ . Note that the differential equations of the original period P ,

$$\begin{aligned} (\Theta_x - A_1) P &= 0, \\ (\Theta_y - A_2) P &= 0, \end{aligned} \tag{11}$$

where both matrices A_1, A_2 have non-vanishing elements only at the lower triangular blocks, the diagonal blocks and the first upper triangular blocks from the diagonal blocks,

$$A_i := \begin{pmatrix} * & * & & & & & O \\ * & * & * & & & & \\ * & * & * & * & & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & & \\ * & * & * & \dots & * & * & \\ * & * & * & \dots & * & * & * \\ * & * & * & \dots & * & * & * \end{pmatrix}. \tag{12}$$

This can be easily understood from the forms of the original differential equations. When we carry out the sweeping-out method on the equations (11), the period matrix P is changed into the Φ by multiplying some lower triangular matrix g ,

$$\Phi = g P,$$

$$g = \begin{pmatrix} * & & & & & & O \\ * & * & & & & & \\ * & * & * & & & & \\ \vdots & \vdots & \vdots & \ddots & & & \\ * & * & * & \cdots & * & & \\ * & * & * & \cdots & * & * & \\ * & * & * & \cdots & * & * & * \end{pmatrix} . \quad (13)$$

Then equations (11) are rewritten as,

$$\left[\Theta_i - (g A_i g^{-1} - g \Theta_i g^{-1}) \right] \Phi = 0 , \quad (i = 1, 2; \Theta_1 := \Theta_x, \Theta_2 := \Theta_y) . \quad (14)$$

That is to say, the matrices A_i are transformed as components of some gauge connection A ,

$$A := A_1 d \log x + A_2 d \log y ,$$

and the transformation by g can be regarded as a sort of gauge transformation. From the forms of A_i and g (12,13), we find that the form of $A_i^g := g A_i g^{-1} - g \Theta_i g^{-1}$ have the similar form as the A_i ,

$$A_i^g := \begin{pmatrix} * & * & & & & & O \\ * & * & * & & & & \\ * & * & * & * & & & \\ \vdots & \vdots & \vdots & \ddots & & & \\ * & * & * & \cdots & * & * & \\ * & * & * & \cdots & * & * & * \\ * & * & * & \cdots & * & * & * \end{pmatrix} . \quad (15)$$

On the other hand, the resulting period matrix Φ has the upper triangular form and the matrix $\Theta_i \Phi$ is also some upper triangular matrix,

$$\Theta_i \Phi = \begin{pmatrix} 0 & * & * & * & \cdots & * & * \\ & 0 & * & * & \cdots & * & * \\ & & 0 & * & \cdots & * & * \\ & & & \ddots & \ddots & \vdots & \vdots \\ & & & & 0 & * & * \\ & & & & & 0 & * \\ O & & & & & & 0 \end{pmatrix} .$$

Comparing the both hand sides of the equations,

$$\Theta_i \Phi = A_i^g \Phi ,$$

we find that the A_i^g must have the following form,

$$A_i^g = \begin{pmatrix} 0 & * & & & O \\ & 0 & * & & \\ & & 0 & * & \\ & & & \ddots & \ddots \\ & & & & 0 & * \\ O & & & & & 0 \end{pmatrix}. \quad (16)$$

Lastly we change the variables (x, y) into the variables (t, s) defined in (10), we obtain the results,

$$\begin{aligned} \partial_{t_i} \Phi &= \kappa_i \Phi, \quad (i = 1, 2; t_1 := t, t_2 := s) \\ \kappa_i &= \partial_{t_i} F, \\ F &:= \begin{pmatrix} 0 & \omega_1^{(\bullet)} & & & O \\ & 0 & \widehat{M}_2^{(1)} & & \\ & & 0 & \widehat{M}_3^{(2)} & \\ & & & \ddots & \ddots \\ & & & & 0 & \widehat{M}_{d-1}^{(d-2)} \\ O & & & & & 0 & u^{(d-1)} \end{pmatrix}. \end{aligned} \quad (17)$$

That is to say, three point couplings $\kappa_i^{(l)}$ defined by fusions,

$$\mathcal{O}_i^{(1)} \cdot \mathcal{O}_j^{(l)} = \left(\kappa_i^{(l)} \right)_j^k \mathcal{O}_k^{(l+1)},$$

are obtained,

$$\left(\kappa_i^{(l)} \right)_j^k = \left(\partial_{t_i} \widehat{M}_{l+1}^{(l)} \right)_{jk}.$$

Because $(l, l+1)$ -th block can be represented as,

$$\widehat{M}_{l+1}^{(l)} = \begin{pmatrix} t & s \\ 0 & t \end{pmatrix} + A_0^{(l)}, \quad (l = 1, 2, \dots, d-2),$$

we get three point functions in the matrix forms,

$$\begin{aligned} \kappa_t^{(l)} &:= \partial_t \widehat{M}_{l+1}^{(l)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \partial_t A_0^{(l)}, \\ \kappa_s^{(l)} &:= \partial_s \widehat{M}_{l+1}^{(l)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \partial_s A_0^{(l)}. \end{aligned}$$

These couplings are represented graphically,

$$\begin{aligned} \kappa_t^{(l)} = & \begin{array}{c} \mathcal{O}_t^{(1)} \\ \bullet \\ | \\ \mathcal{O}^{(l)} \bullet \text{---} \mathcal{O}_s^{(1)} \text{---} \bullet \mathcal{O}^{(l+1)} \\ \bullet \end{array} , \\ \kappa_s^{(l)} = & \begin{array}{c} \bullet \\ | \\ \mathcal{O}^{(l)} \bullet \text{---} \mathcal{O}_s^{(1)} \text{---} \bullet \mathcal{O}^{(l+1)} \\ \bullet \end{array} , \end{aligned}$$

where the symbols $\mathcal{O}_t^{(1)}$, $\mathcal{O}_s^{(1)}$ stand for the injection of some charge $(1, 1)$ operators associated with the Kähler forms J_t, J_s coupled with t, s respectively. In the above figure, charge (l, l) operators $\mathcal{O}^{(l)}$ and the charge $(1, 1)$ operators $\mathcal{O}_{t,s}^{(1)}$ are injected and fuse together. Then some resulting operators $\mathcal{O}^{(l+1)}$ with charge $(l+1, l+1)$ are constructed. In this case, there are just two operators with each definite charge (l, l) ($l = 1, 2, \dots, d-1$). Thus we can represent the couplings which have one of the fixed charge $(1, 1)$ operators $\mathcal{O}_t^{(1)}$, $\mathcal{O}_s^{(1)}$ as 2 by 2 matrices $(\kappa_t^{(l)})$, $(\kappa_s^{(l)})$ ($l = 1, 2, \dots, d-1$). After straightforward calculations, we obtain series expansions of these couplings explicitly,

$$\begin{aligned} \kappa_t^{(l)} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + q_1 \begin{pmatrix} \tilde{a}_1^{(l+1)} & \tilde{a}_2^{(l)} \\ 0 & \tilde{a}_1^{(l)} \end{pmatrix} + O(q^2) , \\ \kappa_s^{(l)} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + q_2 \begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix} + O(q^2) , \\ \tilde{a}_1^{(l)} &:= (d+1)! \cdot \left[- \left(\sum_{n=2}^{d+1} \frac{d+1}{n} \right) \right. \\ &\quad \left. + \sum_{1 \leq m_1 < m_2 < \dots < m_l \leq d} \frac{d+1-m_1}{m_1} \cdot \frac{d+1-m_2}{m_2} \cdot \dots \cdot \frac{d+1-m_l}{m_l} \right] , \\ \tilde{a}_2^{(l)} &:= (d+1)! \cdot 2 \left[-1 + \sum_{1 \leq m_1 < m_2 < \dots < m_l \leq d} \frac{d+1-m_1}{m_1} \cdot \frac{d+1-m_2}{m_2} \cdot \dots \cdot \frac{d+1-m_l}{m_l} \right] , \\ &\quad (l = 1, 2, \dots, d-1) , \\ q_1 &:= \exp(2\pi i t) , \quad q_2 := \exp(2\pi i s) . \end{aligned}$$

The coefficients in the series expansions of these three point functions,

$$\begin{aligned} (\kappa_a^{(l-1)})_{bc} &:= \langle \mathcal{O}_a^{(1)}(0) \mathcal{O}_b^{(l-1)}(1) \mathcal{O}_c^{(d-l)}(\infty) \rangle \\ &= \langle \phi[e_1^{(a)}](0) \phi[e_{l-1}^{(b)}](1) \phi[e_{d-l}^{(c)}](\infty) \rangle , \end{aligned}$$

correspond to the number of holomorphic maps X^i with the conditions,

$$\begin{aligned}\mathcal{O}_a^{(m)}(P) &\equiv \phi[e_m^{(a)}](P) \ , \ e_m^{(a)} \in H_d^{m,m} \ , \ P \in \Sigma \ , \\ X(0) &\in P.D.(e_1^{(a)}) \ , \\ X(1) &\in P.D.(e_{l-1}^{(b)}) \ , \\ X(\infty) &\in P.D.(e_{d-l}^{(c)}) \ ,\end{aligned}$$

where the ϕ maps cohomology elements e_m of M to some $A(M)$ -model operators $\mathcal{O}^{(m)}$. The previous matrix F (17) can be thought of as a sort of a generating function of the three point functions κ_i . This result is quite fascinating. All we have done seem to be a sort of the Miura transformation in two-parameter case. From that viewpoint, the resulting period matrix Φ is the positive root part of the matrix P in the usual Gauss decomposition, and the three point couplings are associated to the Cartan parts of the Gauss decomposition. We make a remark. For the Calabi-Yau three-folds, the three point functions have charge one fields only, and they are symmetric with respect to these three indices. For that reason, one can integrate this function F more twice only for the three-folds.

Next let us consider integrable conditions. Note that we have all explicit forms of solutions of the equations and know all components of the period matrix exactly in series formula. Obviously integrable conditions should exist,

$$[\partial_{t_i} - \kappa_i, \partial_{t_j} - \kappa_j] = 0 \ . \quad (18)$$

From this relation (18) and the equation $\kappa_i = \partial_i F$, we have some sort of associativities among these couplings,

$$[\kappa_i, \kappa_j] = 0 \ . \quad (19)$$

They can be rewritten in components of the matrices κ_i ($i = 1, 2$) as,

$$\kappa_{t_i}^{(l)} \kappa_{t_j}^{(l+1)} - \kappa_{t_j}^{(l)} \kappa_{t_i}^{(l+1)} = 0 \ .$$

This relation tells us that any correlation functions are independent of the positions of the insertion of the charge one operators $\mathcal{O}_j^{(1)}$ and this property is reasonable from the physical point of view. Now we consider d-point functions of charge (1, 1) operators $\mathcal{O}_t^{(1)}$, $\mathcal{O}_s^{(1)}$,

$$\begin{aligned} &K_{t_{i_1} t_{i_2} \dots t_{i_{d-1}} t_{i_d}} \\ &= \langle \mathcal{O}_{t_{i_1}}^{(1)} \mathcal{O}_{t_{i_2}}^{(1)} \dots \mathcal{O}_{t_{i_{d-1}}}^{(1)} \mathcal{O}_{t_{i_d}}^{(1)} \rangle \\ &= \left(\kappa_{t_{i_2}}^{(1)} \kappa_{t_{i_3}}^{(2)} \kappa_{t_{i_4}}^{(3)} \dots \kappa_{t_{i_{d-2}}}^{(d-3)} \kappa_{t_{i_{d-1}}}^{(d-2)} \eta^{(d-1)} \right)_{t_{i_1} t_{i_d}} \end{aligned}$$

==

metric),

$$\eta^{(d-1)} := \langle \mathcal{O}^{(d-1)} \mathcal{O}^{(1)} \rangle = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

matrix form,

$$=$$

We find these d-point couplings,

$$\begin{aligned}
K_{d,0} &= 2(d+1) \\
&\quad + 2(d+1) \cdot \left\{ 2 \cdot (d+1)^{(d+1)} - (d+2) \cdot (d+1)! - d \cdot (d+1)! \left(\sum_{m=2}^{d+1} \frac{d+1}{m} \right) \right\}_{q_1} + O(q^2) \quad , \\
K_{d-1,1} &= (d+1) \\
&\quad + (d+1) \cdot \left\{ (d+1)^{(d+1)} - 2 \cdot (d+1)! - (d-1) \cdot (d+1)! \left(\sum_{m=2}^{d+1} \frac{d+1}{m} \right) \right\}_{q_1} + O(q^2) \quad , \\
K_{d-2,2} &= 0 + O(q^2) \quad , \\
K_{d-3,3} &= (d+1)q_2 + O(q^2) \quad , \\
K_{d-n,n} &= 0 + O(q^2) \quad , \quad (n \geq 4) \quad .
\end{aligned}$$

In this article, we treated the one-parameter models and the two-parameter model concretely, but the method developed in this article is not restricted to these cases only.

5 Conclusion

In this article, we have investigated some properties of the higher dimensional Calabi-Yau manifolds subject to the assumption of the existence of the mirror symmetries. We extend the method of calculating the three point functions for the one-parameter families of d -folds and a two-parameter family of d -fold for Calabi-Yau cases. The recipe developed here can be applied to more general cases (for instance the complete intersection Calabi-Yau d -folds in the toric cases). Explicit forms of the homology cycles are not available, but the monodromy properties of the period matrix illustrate the correctness of our Ansatz. We used the mirror conjecture and our results should be verified by the mathematical methods in enumerative geometry [24].

For the general Kähler manifolds M , the virtual dimension of the $A(M)$ -model has a term depending on the first Chern class and the degree of maps. Because of the existence of this term, the degree of the observables are defined modulo $c_1(M)$. (There exists each topological selection rule when one fixes the degree of maps). Also from the point of view of the deformation of the topological field theories, we perturb the topological theories by adding only operators associated with the Kähler forms of M in our cases. In these situations, the charge conservation in each fusion of operators can be discussed almost classically. Only difference between the $A(M)$ -model with $c_1(M)=0$ and the one with $c_1(M)\neq 0$ is that the former has nilpotent structures of operators $\mathcal{O}^{(1)}\mathcal{O}^{(d)}=0$ but the latter does not have these properties. Also the fusion couplings of operators in the former cases have contributions from all degrees of maps because the virtual dimension is independent of the degree of maps. That is to say, the three-point couplings in the Calabi-Yau cases are infinite series with respect to indeterminates $q_l := e^{2\pi i t_l}$ ($l = 1, 2, \dots, \dim H^{1,1}(M)$) associated with a set of Kähler coordinates $\{t_l\}$.

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